Hankel Transform :-

Defn

if $0 < r < \infty$, the Hankel transform of the function $f(r)$ is define as $\bar{f}(p) = H\{f(r), p\} = \int_0^\infty f(r)\, r\, J_n(pr)\, dr$

Where $J_n(pr)$ is the Bessel func of the 1st kind of order n and is denoted by $H\{f(r); p\}$ or $\bar{f}(p)$

$r J_n(pr)$ is the kernel of the transformation.

Inversion formula of Hankel transformation

if $\bar{f}(p)$ is Hankel transform of the func $f(r)$ that is $\bar{f}(p) = \int_0^\infty f(r)\, r\, J_n(pr)\, dr$

then

$$f(r) = \int_0^\infty \bar{f}(p)\, p\, J_n(pr)\, dp$$

is called the inversion formula for the Hankel transform $\bar{f}(p)$.

An we write $f(r) = H^{-1}\{\bar{f}(p), r\}$.

Some important result for Bessel function:-

I. The Bessel func of 1st kind :-

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

II. The recurrence relations for $J_n(x)$:-

(a) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(b) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

(3) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(4) $2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

⑤ $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}$

⑥ $\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x)$

Linearity property of Hankel funⁿ :-
if $f(x)$ and $g(x)$ are two functions and a, b are two constant then

$$H\{a f(x) + b g(x)\} = a H\{f(x)\} + b H\{g(x)\}$$

~~Elementary property of the~~
Infinite integral involving Bessel funⁿ :-

(i) $$\int_0^\infty e^{-ax} J_0(px)\, dx = \frac{1}{(a^2+p^2)^{\frac{1}{2}}}$$

(ii) $$\int_0^\infty e^{-ax} J_1(px)\, dx = \frac{1}{p} - \frac{a}{p(a^2+p^2)^{\frac{1}{2}}}$$

(iii) $$\int_0^\infty x e^{-ax} J_0(px)\, dx = a(a^2+p^2)^{-3/2}$$

(iv) $$\int_0^\infty x e^{-ax} J_1(px)\, dx = p\,(a^2+p^2)^{3/2}$$

(v) $$\int_0^\infty \frac{1}{x} e^{-ax} J_1(px)\, dx = \frac{(a^2+p^2)^{\frac{1}{2}} - a}{p}$$

Elementary property of Hankel transform

i) Hankel transform of nth order of function of the type $f(ax)$ is given by

$$H_n\{f(ax); p\} = \int_0^\infty x\, f(ax)\, J_n(px)\, dx, \quad a>0$$

put $r = ax$

$dr = a\,dx$

$$H_n\{f(ax); p\} = \int_0^\infty \frac{r}{a} f(r)\, J_n\left(\frac{pr}{a}\right) \frac{dr}{a}$$

$$= \frac{1}{a^2}\int_0^\infty r\, f(r)\, J_n\left(\frac{p}{a} r\right) dr$$

$$\boxed{H_n\{f(ax); p\} = a^{-2} H_n\left\{f(r); \frac{p}{a}\right\}} \quad (1)$$

ii) let us consider the recurrence relation

(iv) for Bessel func

$$\cancel{2n J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]}$$

$$\cancel{\int_0^\infty x e^{-ax} J_1(px)\, dx = p(a^2+p^2)^{-\frac{3}{2}}}$$

$$\Rightarrow \cancel{J_{n-1}(x) - \frac{2n}{x} J_n(x) + J_{n+1}(x) = 0}$$

Substituting $p(r)$ for x and multiply both sides of resulting eqns by $r f(r)$ and integrating with $(0,\infty)$ w.r.t r we get –

$$2n\int_0^\infty \frac{1}{pr}\, r f(r)\, J_n(pr)\, dr = \int_0^\infty r f(r) J_{n-1}(pr)\, dr + \int_0^\infty r f(r) J_{n+1}(pr)\, dr$$

$$\int_0^\infty f(r) J_n(pr)\,dr = \frac{p}{2n}\left[\int_0^\infty r f(r) J_{n-1}(pr)\,dr + \int_0^\infty r f(r) J_{n+1}(pr)\,dr\right]$$

$$H_n\{r^{-1} f(r); p\} = \frac{p}{2n}\left[\bar{f}_{n-1}(p) + \bar{f}_{n+1}(p)\right]$$

for $n \neq 0$

Problem:-

1) find the Hankel transform of

$$f(x) = \begin{cases} 1, & 0 < x < a, \text{ for } n = 0 \\ 0, & x > a \end{cases}$$

Soln: We know that

$$H_n\{f(x); p\} = \int_0^\infty f(x)\, x J_n(px)\,dx$$

When $n = 0$ we have

$$H_0\{f(x)\} = \int_0^\infty f(x)\, x J_0(px)\,dx$$

$$= \int_0^a f(x)\, x J_0(px)\,dx$$

$$= \int_0^a x J_0(px)\,dx \quad \text{—— (1)}$$

from recurrence formula of Bessel func

From 6

$\frac{d}{dx}[x^n J$

putting

$\frac{d}{dx}[$

now wr

$\Rightarrow$

$\frac{1}{p}\int_0^a \frac{d}{dx}[$

$\Rightarrow \frac{1}{p}$

$\Rightarrow \frac{1}{p}$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

from 6

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

putting n=1 we get –

$$\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)$$

now writing p(x) for x we get

$$\Rightarrow \frac{1}{p}\frac{d}{dx}\left[p(x) J_1(p(x))\right] = p(x) J_0(p(x))$$

$$\frac{1}{p}\int_0^a d\left[px J_1(px)\right] = \int_0^a px J_0(px)\, dx$$

$$\Rightarrow \frac{1}{p}\left[px J_1(px)\right]_0^a = \int_0^a (px) J_0(px)\, dx$$

$$\Rightarrow \frac{1}{p}\left[pa J_1(ap)\right] = \int_0^a (px) J_0(px)\, dx$$

$$\Rightarrow a J_1(ap) = \int_0^a (px) J_0(px)\, dx$$

$$\Rightarrow \int_0^a x J_0(px)\, dx = \frac{a}{p} J_1(pa)$$

$$\Rightarrow \boxed{H_0\{f(x)\} = \frac{a}{p} J_1(pa)}$$

(from ①)

2) Find the Hankel transform of $x^{-2}e^{-x}$ taking $xJ_1(px)$ as the kernal.

$$H\{x^{-2}e^{-x};p\} = \int_0^\infty x^{-2}e^{-x} \cdot xJ_1(px)\,dx$$

$$= \int_0^\infty \frac{1}{xe^x} J_1(px)\,dx$$

$$= \int_0^\infty \frac{1}{x} e^{-x} J_1(px)\,dx$$

$$= \frac{(1+p^2)^{\frac{1}{2}} - 1}{p}$$

[from ⑤]

3) Find the Hankel transform of

$$f(x) = \begin{cases} x^n & ; 0 < x < a \\ 0 & ; x > a \end{cases}$$

taking $xJ_n(px)$ as the kernal

4) Find the Hankel transform of $\frac{1}{r}$ and find the original func for $n=0$

(find $f(r)$)

3) $f(x) = \begin{cases} x^n & 0 < x < a \\ 0 & x > a \end{cases}$

we know that

$$H_n\{f(x); p\} = \int_0^\infty f(x)\, x\, J_n(px)\, dx$$

$$H_n\{f(x)\} = \int_0^a x^n \cdot x\, J_n(px)\, dx$$

$$= \int_0^a x^{n+1} J_n(px)\, dx \quad \text{——— (1)}$$

$\boxed{\text{putting } \cancel{n = n-1} \text{ we } \cancel{\text{have}} \quad H_{n-1}\{f(\cancel{x})\} = \cancel{\int_0^a x^n\, J_{n-1}}}$

we know that from reccurence relation

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

now writing $p(x)$ for x

$$\frac{1}{p}\int_0^a \frac{d}{dx}\left[(px)^n J_n(px)\right] = \int_0^a (px)^n J_{n-1}(px)\, dx$$

$$\Rightarrow \frac{1}{p}\left[p^n a^n J_n(pa)\right]_0^a = p^n \int_0^a x^n J_{n-1}(px)\, dx$$

$$\Rightarrow \int_0^a x^n J_{n-1}(px)\, dx = \frac{a^n}{p} J_n(pa)$$

now we replace n by $n+1$ we get

$$\int_0^a x^{n+1} J_n(px)\, dx = \frac{a^{n+1}}{p} J_{n+1}(pa)$$

now from (1)

$$\boxed{H_n\{f(x)\} = \frac{a^{n+1}}{p} J_{n+1}(pa)}$$

4) $H_n\{f(r), p\} = \int_0^\infty \frac{1}{r} \cdot r J_n(pr)\, dr$

$H_n\{f(r)\} = \int_0^\infty J_n(pr)\, dr$

$\bar{f}(p) = \int_0^\infty J_n(pr)\, dr$

$f(r) = \int_0^\infty \bar{f}(p)\, p J_n(pr)\, dp$

$\bar{f}(p) = \int_0^\infty J_0(pr)\, dr$

$n = 0$

① $\bar{f}(p) = \int_0^\infty J_0(pr)\, dr$

from ⑥

$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$\frac{d}{dx}\left[J_0(x)\right] = J_{-1}(x)$

now taking px for x we get

$\frac{1}{p}\frac{d}{dx}\left[J_0(px)\right] = J_{-1}(px)$

$\Rightarrow \frac{1}{p}$

4) $H\{f(r), p\} = \bar{f}(p) = \int_0^\infty \frac{1}{r} \cdot r J_0(pr)\, dr$

$\int_0^\infty e^{-ar} J_0(pr)\, dr$ $\quad [a = 0]$

$\int_0^\infty e^{-ar} J_0(pr)\, dr = (a^2 + p^2)^{-\frac{1}{2}}$

$= (a^2 + p^2)^{-\frac{1}{2}}$

$= \frac{1}{p} \quad [\because a = 0$

$f(r) = \int_0^\infty \bar{f}(p) \cdot p J_0(pr)\, dp$

$$= \int_0^\infty \frac{1}{p} \cdot p \, J_0(pr) \, dp$$

$$= \int_0^\infty J_0(pr) \, dp$$

$$= \int_0^\infty e^{-ap} J_0(pr) \, dp$$

$$= (a^2 + r^2)^{-\frac{1}{2}}$$

$$= \frac{1}{r} \quad [\because a = 0]$$

Parseval's theorem for Hankel transforms :-

If $\bar{f}(p)$ and $\bar{g}(p)$ are the Hankel transforms of the functions $f(x)$ & $g(x)$ respectively. Then

$$\int_0^\infty x f(x) g(x) \, dx = \int_0^\infty p \bar{f}(p) \bar{g}(p) \, dp$$

Proof: We have $\int_0^\infty x f(x) J_n(px) \, dx = \bar{f}(p)$

and $\int_0^\infty x g(x) J_n(px) \, dx = \bar{g}(p)$

Now $\int_0^\infty p \bar{f}(p) \bar{g}(p) \, dp = \int_0^\infty p \bar{f}(p) \, dp \int_0^\infty x g(x) J_n(px) \, dx$

$$= \int_0^\infty x g(x) \, dx \left\{ \int_0^\infty p \bar{f}(p) J_n(px) \, dp \right\}$$

$$= \int_0^\infty x f(x) g(x) \, dx$$

(interchanging the order of integration)

(using inversion formula for $f(x)$)

Hankel Transform of derivatives of a function :-

The Hankel transform of order n of the function $f(x)$ is given by

$$H_n\{f(x);p\} = \bar{f}_n(p)$$

$$= \int_0^\infty x f(x) J_n(px)\, dx \quad \text{--- (1)}$$

if $\bar{f}_n'$ is the Hankel transform of $\frac{df}{dx} (= f')$ then we have

$$H_n\{f'(x);p\} = \bar{f}_n'(p) = \int_0^\infty x \frac{df}{dx} J_n(px)\, dx \quad \text{--- (2)}$$

$$= \left[\left(x J_n(px) f(x)\right) - \int_0^\infty f(x) \frac{d}{dx}\left(x J_n(px)\right) dx\right.$$

$$= 0 - \int_0^\infty f(x)\left[J_n(px) + x p J_n'(px)\right] dx \quad \text{--- (3)}$$

assume that $x f(x) \to 0$ when $x \to 0$ and $x \to \infty$

from the recurrence relation from the Bessel's func we get:-

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

let us replace x by px we get

$$px J_n'(px) = -n J_n(px) + px J_{n-1}(px) \quad \text{--- (4)}$$

using (4), (3) becomes

$$\bar{f}_n'(p) = -\int_0^\infty f(x)\left[J_n(px) - n J_n(px) + px J_{n-1}(px)\right] dx$$

$$= -\int_0^\infty f(x)\left[(1-n)J_n(px) + px\,J_{n-1}(px)\right]dx$$

$$= (n-1)\int_0^\infty f(x)\,J_n(px)\,dx - \int_0^\infty px\,J_{n-1}(px)\,f(x)\,dx \quad \text{--- (5)}$$

Now we use the following recurrence relations for Bessel's functions —

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

replacing x by px we get —

$$J_n(px) = \frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right] \quad \text{--- (6)}$$

using (6), (5) becomes —

$$= (n-1)\int_0^\infty f(x)\cdot\frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right]dx$$

$$- \int_0^\infty px\,f(x)\,J_{n-1}(px)\,dx$$

$$= \int_0^\infty \left\{(n-1)\frac{px}{2n} - px\right\} f(x)\,J_{n-1}(px)\,dx$$

$$+ \int_0^\infty \frac{(n-1)px}{2n}\,f(x)\,J_{n+1}(px)\,dx$$

$$= -\frac{p(n+1)}{2n}\int_0^\infty x\,f(x)\,J_{n-1}(px)\,dx + \frac{p(n-1)}{2n}\int_0^\infty x\,f(x)\,J_{n+1}(px)\,dx$$

$$\bar{f}_n'(p) = -p\left[\frac{n+1}{2n}\bar{f}_{n-1}(p) - \frac{n-1}{2n}\bar{f}_{n+1}(p)\right] \quad (7)$$

To obtain the Hankel transform of 2nd derivative. Let us replace f by f' in (7) and we get

$$\bar{f}_n''(p) = -p\left[\frac{n+1}{2n}\bar{f}_{n-1}'(p) - \frac{n-1}{2n}\bar{f}_{n+1}'(p)\right] \quad (8)$$

replacing n by $n-1$ and $n+1$ respectively in (7) we get —

$$\bar{f}_{n-1}'(p) = -p\left[\frac{n}{2(n-1)}\bar{f}_{n-2}(p) - \frac{n-2}{2(n-1)}\bar{f}_n(p)\right] \quad (9)$$

$$\bar{f}_{n+1}'(p) = -p\left[\frac{n+2}{2(n+1)}\bar{f}_n(p) - \frac{n}{2(n+1)}\bar{f}_{n+2}(p)\right] \quad (10)$$

Substituting (9) & (10) in (8) we get

$$\bar{f}_n''(p) = -p\left[\frac{n+1}{2n}\left(-p\left\{\frac{n}{2(n-1)}\bar{f}_{n-2}(p) - \frac{n-2}{2(n-1)}\bar{f}_n(p)\right\}\right)\right.$$

$$\left. -p\left[-\frac{n-1}{2n}\left\{-p\left(\frac{n+2}{2(n+1)}\bar{f}_n(p) - \frac{n}{2(n+1)}\bar{f}_{n+2}(p)\right)\right\}\right]\right]$$

$$\bar{f}_n''(p) = \frac{p^2}{4}\left[\frac{n+1}{n-1}\bar{f}_{n-2}(p) + \frac{n-1}{n+1}\bar{f}_{n+2}(p) - 2\frac{n^2-3}{n^2-1}\bar{f}_n(p)\right]$$

$$-\frac{(n-2)(n+1)}{n-1} - \frac{(n-1)(n+2)}{n+1}$$

$$= -(n-2)\frac{(n+1)(n+1) + (n-1)^2}{(n-1)(n+1)}$$

$$= -(n-2)\frac{[n^2+2n+1 + n^2-2n+1]}{n^2-1}$$

$$= -2(n-2)\frac{(n^2+1)}{n^2-1}$$

$$= \frac{-2}{n^2-1}(n^3+n-2n^2-2)$$

Some Reduction :-

putting $n=1$ in (7) we get

$$\bar{f}_1' = -p\,\bar{f}_0(p)$$

$$= H_1\{f'(x);p\}_0$$

putting $n=2$ in (7) we get -

$$\bar{f}_2' = -p\left[\frac{3}{4}\bar{f}_1(p) - \frac{1}{4}\bar{f}_3(p)\right]$$

putting $n=3$ in (7) we get -

$$\bar{f}_3' = -p\left[\frac{2}{3}\bar{f}_2(p) - \frac{1}{3}\bar{f}_4(p)\right]$$

Hankel transform of

$$\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f$$

We have $H\left\{\frac{d^2f}{dx^2};p\right\}_0 = \int_0^\infty \frac{d^2f}{dx^2}\,x J_n(px)\,dx$

$$= \left[x J_n(px)\frac{df}{dx}\right]_0^\infty - \int_0^\infty \frac{df}{dx}\frac{d}{dx}\{x J_n(px)\}\,dx$$

(5)

$$= 0 - \int_0^\infty \frac{df}{dx}\left[J_n(px) + xpJ_n'(px)\right]dx$$

assuming that $f'(x) \to 0$ as $x \to 0$

and $x \to \infty$

$$\int_0^\infty \frac{d^2f}{dx^2} x J_n(px)\,dx + \int_0^\infty J_n(px)\frac{df}{dx}\,dx$$

$$= -p\int_0^\infty \frac{df}{dx}\, x J_n'(px)\,dx$$

$$\int_0^\infty \frac{d^2f}{dx^2} x J_n(px)\,dx + \int_0^\infty \frac{1}{x} x J_n(px)\frac{df}{dx}\,dx$$

$$= -p\int_0^\infty \frac{df}{dx}\, x J_n'(px)\,dx \quad (1)$$

$$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right) x J_n(px)\,dx = -p\int_0^\infty \frac{df}{dx} x J_n'(px)\,dx$$

Now R.H.S

$$-p\int_0^\infty \frac{df}{dx} x J_n'(px)\,dx$$

$$= -p\left[f(x)\, x J_n'(px)\right]_0^\infty$$

$$+ p\int_0^\infty f(x)\frac{d}{dx}\left[x J_n'(px)\right]dx$$

$$= p\int_0^\infty f(x)\left[\cancel{J_n'(px)} + \frac{d}{dx}(x J_n'(px))\right]dx$$

assuming $x \to 0$ & $x \to \infty$ as $x f(x) \to 0$

Since

$J_n(x)$ satisfies Bessel's equations

$$\boxed{\frac{d}{dx}\left\{x\frac{dy}{dx}\right\} + \left(1-\frac{n^2}{x^2}\right)xy = 0}$$

we get — $\underline{y = J_n(x)}$

$$\frac{d}{dx}\left\{x\frac{d}{dx}J_n(x)\right\} + \left(1-\frac{n^2}{x^2}\right)xJ_n(x) = 0$$

Now replacing x by px in the above eqns we get —

$$\frac{1}{p}\frac{d}{dx}\left\{\frac{(px)}{p}\frac{d}{dx}J_n(px)\right\} + \left(1-\frac{n^2}{p^2x^2}\right)pxJ_n(px) = 0$$

$$\Rightarrow \frac{1}{p}\frac{d}{dx}\left\{px\, J_n'(px)\right\} + \left(1-\frac{n^2}{p^2x^2}\right)px\, J_n(px) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x J_n'(px)\right] = -\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{p}J_n(px) \quad \text{——— (3)}$$

Now from (1), (2) & (3) we get —

$$\int_0^\infty \left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)xJ_n(px)\,dx \quad \text{——— (1)}$$

$$(2) = -p\int_0^\infty f(x)\left[\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{p}J_n(px)\right]dx$$

$$\Rightarrow \int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f\right)xJ_n(px)\,dx$$

$$= -p^2\int_0^\infty f(x)\,xJ_n(px)\,dx$$

$$= -p^2\bar{f}_n(p) = -p^2 H_n\{f(x)\}$$

Inverse transformation

Statement :-

If $\bar{f}(p)$ is the Hankel transform of the function $f(r)$ that is $\bar{f}(p) = \int_0^{\infty} f(r)\, r\, J_n(pr)\, dr$ then

$$f(r) = \int_0^{\infty} \bar{f}(p)\, p\, J_n(pr)\, dp$$

Proof

if $\bar{f}(s)$ denotes the complex fourier transformation of $f(x)$ that is $\bar{f}(s)$

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \;\longrightarrow\; -\infty < x < \infty$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{-isx}\, ds$$

extending this result for the func of two variables we have

$$\bar{f}(s,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, e^{i(sx+ty)}\, dx\, dy \quad \text{——— (1)}$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(s,t)\, e^{-i(sx+ty)}\, ds\, dt \quad \text{——— (2)}$$

Now putting $x = r\cos\theta$
$y = r\sin\theta$
$s = p\cos\alpha$
$t = p\sin\alpha$ *

in (1) & (2) we get =

If $\bar{f}(p,\alpha) = \int_0^\infty r\,dr \int_0^{2\pi} f(r,\theta)\, e^{irp\cos(\theta-\alpha)}\, d\theta$ —— (3)

then

$f(r,\theta) = \frac{1}{4\pi^2}\int_0^\infty p\,dp \int_0^{2\pi} \bar{f}(p,\alpha)\, e^{-irp\cos(\theta-\alpha)}\, d\alpha$ —— (4)

let us suppose

$f(r,\theta)$ is a special function $f(r)e^{-in\theta}$

then (3) becomes →

$$\bar{f}(p,\alpha) = \int_0^\infty f(r) r\,dr \int_0^{2\pi} e^{i\{-n\theta + rp\cos(\theta-\alpha)\}}\, d\theta$$

taking $\phi = \alpha - \theta - \frac{\pi}{2}$

$$\bar{f}(p,\alpha) = \int_0^\infty f(r)\, r\, dr \int_0^{2\pi} e^{i\{n(\phi-\alpha+\frac{\pi}{2}) + rp\cos(\phi+\frac{\pi}{2})\}}\, d\phi$$

$$= \int_0^\infty r f(r)\, dr\; e^{in(\frac{\pi}{2}-\alpha)} \int_0^{2\pi} e^{i(n\phi - rp\sin\phi)}\, d\phi$$

$$= \int_0^\infty r f(r)\, e^{in(\frac{\pi}{2}-\alpha)} \cdot 2\pi J_n(pr)\, dr$$

$\left[\because J_n(pr) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\phi - rp\sin\phi)}\, d\phi\right]$

$$= 2\pi e^{in(\frac{\pi}{2}-\alpha)} \int_0^\infty r f(r)\, J_n(pr)\, dr$$

$$= 2\pi e^{in(\frac{\pi}{2}-\alpha)}\, \bar{f}(p)$$ —— (5)

where $\bar{f}(p) = \int_0^\infty r f(r)\, J_n(pr)\, dr$

now putting

$f(r,\theta) = f(r)\, e^{-in\theta}$

$\bar{f}(p,\alpha) = 2\pi\, e^{in(\frac{\pi}{2}-\alpha)}\, \bar{f}(p)$

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in (i) we get

$$f(r)\, e^{-in\theta} = \frac{1}{4\pi^2}\int_0^\infty p\,dp \int_0^{2\pi} 2\pi\, e^{in(\frac{\pi}{2}-\alpha)}\, \bar{f}(p)\, e^{-irp\cos(\theta-\alpha)}\, d\alpha$$

$$= \frac{1}{2\pi}\int_0^\infty p\,\bar{f}(p)\,dp \int_0^{2\pi} e^{i\{n(\frac{\pi}{2}-\alpha) - rp\cos(\theta-\alpha)\}}\, d\alpha$$

$$= \frac{1}{2\pi}\int_0^\infty p\,\bar{f}(p)\,dp \int_0^{2\pi} e^{i\{n(\psi-\theta) - rp\cos(\frac{\pi}{2}-\psi)\}}\, d\psi$$

where $\psi = \theta - \alpha + \frac{\pi}{2}$

$$= \frac{1}{2\pi}\int_0^\infty p\,\bar{f}(p)\, e^{-in\theta}\, dp \int_0^{2\pi} e^{i\{n\psi - rp\sin\psi\}}\, d\psi$$

$$= \frac{1}{2\pi}\int_0^\infty p\,\bar{f}(p)\, e^{-in\theta}\; 2\pi\, J_n(pr)\, dp$$

$$= e^{-in\theta} \cdot \cancel{\frac{2\pi}{2\pi}} \int_0^\infty p\,\bar{f}(p)\, J_n(pr)\, dp$$

$$\Rightarrow f(r) = \int_0^\infty p\,\bar{f}(p)\, J_n(pr)\, dp$$

1) Find the Hankel transformation of

$$f(x) = \begin{cases} a^2 - x^2 & 0 < x < a, \; n = 0 \\ 0, & x > a \quad n = 0 \end{cases}$$

We know that

$$H_n\{f(x); p\} = \int_0^\infty f(x)\, x\, J_n(px)\, dx$$

$$= \int_0^a (a^2 - x^2)\, x\, J_0(px)\, dx \qquad [\because n = 0]$$

$$= a^2 \int_0^a x J_0(px)\, dx - \int_0^a x^3 J_0(px)\, dx \quad \text{--- (1)}$$

now

$\cancel{= a^2 \frac{a}{p} J_1(pa)}$

$$\int_0^a x J_0(px)\, dx$$

$$= \frac{a}{p} J_1(pa) \quad \text{--- (2)}$$

now

for the recurrence formula (6) for $J_n(x)$ we get

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

writing $p(x)$ for x, and $n = 1$ in above formula we have

$$\frac{1}{p}\frac{d}{dx}\left[(px) J_1(px)\right] = px\, J_0(px)$$

$$\frac{d}{dx}\left[x J_1(px)\right] = px\, J_0(px)$$

H.W. Find the Hankel transform of e^{-ax^2} taking $x J_0(px)$ as the kernel of the transform

must do: $f(x) = 1 \quad 0 < x < a$; $\quad 0 \quad x > 0$

$\Rightarrow \frac{1}{p}\frac{d}{dx}\left[x J_1(px)\right] = x J_0(px)$

therefore,

$$\int_0^a x^3 J_0(px)\,dx = \int_0^a x^2 \left(x J_0(px)\right) dx$$

$$= \int_0^a x^2 \, \frac{1}{p}\frac{d}{dx}\left[x J_1(px)\right] dx$$

$$= \frac{1}{p}\left[x^2 \left\{x J_1(px)\right\}\right]_0^a$$

$$- \frac{2}{p}\int_0^a x \cdot x J_1(px)\, dx \quad \text{---} (3)$$

Again writing $p(x)$ for x and $n=2$ in the recurrence formula (6) we get

$$\frac{1}{p}\frac{d}{dx}\left[x^2 J_2(px)\right] = x^2 J_1(px)$$

therefore from (3)

$$\int_0^a x^3 J_0(px)\,dx$$

$$= \frac{a^3}{p} J_1(pa)$$

$$- \frac{2}{p}\int_0^a \frac{1}{p}\frac{d}{dx}\left\{x^2 J_2(px)\right\} dx$$

$$= \frac{a^3}{p} J_1(pa) - \frac{2}{p^2}\left[x^2 J_2(px)\right]_0^a \quad \text{---} (A)$$

$$= \frac{a^3}{p} J_1(pa) - \frac{2}{p^2}\left[a^2 J_2(pa)\right]$$

from recurrence formula (4b) in the Bessel's func

$2n\, J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

writing $n=1$ and ap for x in the above recurrence formula we get —

$$2J_1(ap) = ap\left[J_0(ap) + J_2(ap)\right]$$

$$\Rightarrow J_2(ap) = \frac{2J_1(ap)}{ap} - J_0(ap) \quad \text{—— (4)}$$

putting (4) in A we get

$$\int_0^a x^3 J_0(px)\,dx$$

$$= \left[\frac{a^3}{p} J_1(pa) - \frac{2a^2}{p^2}\left[\frac{2J_1(ap)}{ap} - J_0(ap)\right]\right]$$

$$= \frac{a^3p^2 - 4a}{p^3} J_1 + \frac{2a^2}{p^2} J_0(ap)$$

now

$$H_n(f(r)) = a^2 \cdot \frac{a}{p} J_1(pa) - \frac{a^3p^2 - 4a}{p^3} J_1 - \frac{2a^2}{p^2} J_0(ap)$$

$$= \frac{4a}{p^3} J_1(pa) - \frac{2a^2}{p^2} J_0(pa)$$

Determination

Evaluate $\int_0^\infty r\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) J_0(pr)\,dr$

$f(r) = \dfrac{e^{-ar}}{r}$

Proof: $H\left\{\frac{d^2f}{dr^2}; p\right\} = \int_0^\infty \frac{d^2f}{dr^2}\, r J_n(pr)\,dr$

$$\left[\frac{dR}{dr}\cdot r\, J_n(pr)\right]_0^\infty - \int_0^\infty \frac{dR}{dr}\cdot\frac{d}{dr}\left[r\, J_n(pr)\right] dr$$

$$= 0 - \int_0^\infty \frac{dR}{dr}\left[J_n(pr) + pr\, J_n'(pr)\right] dr$$

because $r\, J_n(pr) \to 0$ as $r \to \infty$ & $r \to 0$

$\Rightarrow$

$$\int_0^\infty \left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) r\, J_n(pr)\, dr$$

$$= -\int_0^\infty pr\, \frac{dR}{dr}\, J_n'(pr)\, dr$$

now

R.H.S

$$-\int_0^\infty pr\, \frac{dR}{dr}\, J_n'(pr)\, dr = -p\left\{\left[r\, R(r)\, J_n'(pr)\right]_0^\infty - \int_0^\infty R(r)\frac{d}{dr}\{r\, J_n'(pr)\} dr\right.$$

$$= +p\int_0^\infty R(r)\, \frac{d}{dr}\left[r\, J_n'(pr)\, dr\right]$$

assuming ~~Rtom be~~ $r \to 0$ & $r \to \infty$ as $r\, R(r) \to 0$

now we have

$$\int_0^\infty \left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) r\, J_n(pr)\, dr$$

$$= p\int_0^\infty R(r)\, \frac{d}{dr}\left[r\, J_n'(pr)\right] dr$$

putting $n = 0$ we have

$$\int_0^\infty r\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) J_0(pr)\, dr$$

$$= p\int_0^\infty R(r)\, \frac{d}{dr}\left[r\, J_0'(pr)\right] dr$$

now $y = J_n(x)$

$$\frac{d}{dx}\left(x \frac{d\,J_n(x)}{dx}\right) + \left(1 - \frac{n^2}{x^2}\right) x J_n(x) = 0$$

$$\Rightarrow \frac{1}{p}\frac{d}{dx}\left(\frac{px}{p}\frac{d}{dx} J_n(px)\right) + \left(1 - \frac{n^2}{p^2x^2}\right) px J_n(px) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x J_n'(px)\right] = -\left(p^2 - \frac{n^2}{x^2}\right)\frac{x}{p} J_n(px)$$

$\therefore$

$$\int_0^\infty \left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2} f\right) r J_n(pr)$$

$$= -p^2 \int_0^\infty f(r)\, r J_n(pr)\, dr$$

putting $n = 0$

$$\int_0^\infty r\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) J_0(pr)\, dr$$

$$= -p^2 \int_0^\infty \frac{e^{-ar}}{r} \cdot r \cdot J_0(pr)\, dr$$

where given $f(r) = \frac{e^{-ar}}{r}$

$$= -p^2 \int_0^\infty e^{-ar} J_0(pr)\, dr$$

$\cancel{-p^2 a (a^2 + p^2)^{-3/2}}$

$$= \frac{-p^2}{\sqrt{a^2 + p^2}}$$

Problem

1) The vibration of very large membrane are governed by the following equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}, \quad r \geq 0, \; t \geq 0$$

$U = f(r)$, $\frac{\partial U}{\partial t} = g(r)$ where $t = 0$

Show that $t > 0$

$$U(r,t) = \int_0^\infty p\bar{f}(p)\cos(pct)J_0(pr)\,dp + \frac{1}{c}\int_0^\infty \bar{g}(p)\sin(pct)J_0(pr)\,dp$$

where $\bar{f}(p)$ & $\bar{g}(p)$ are the zeroth order Hankel transforms of $f(r)$ & $g(r)$ respectively

Soln

The given partial differential eqn

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \quad \text{--- (1)}$$

taking the Hankel transform for $n = 0$ of both sides eqn (1) and the given conditions we have

$$\int_0^\infty \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) r J_0(pr)\,dr = \frac{1}{c^2}\int_0^\infty \frac{\partial^2 U}{\partial t^2} r J_0(pr)\,dr$$

$$\Rightarrow -p^2 \tilde{U} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_0^\infty U r J_0(pr) dr$$

$$= \frac{1}{c^2} \frac{d^2 \tilde{U}}{dt^2} \quad \left[\text{where } \tilde{U} = \int_0^\infty U r J_0(pr) dr\right]$$

$$\Rightarrow \frac{d^2 \tilde{U}}{dt^2} + p^2 c^2 \tilde{U} = 0 \quad \text{———— (2)}$$

also when $t = 0$, $U = f(r)$ and after using H.T of zeroth order we have

$$\Rightarrow \int_0^\infty U r J_0(pr) dr = \int_0^\infty f(r) r J_0(pr) dr$$

$$\Rightarrow \boxed{\tilde{U} = \tilde{f}(p)} \quad \text{———— (3)}$$

also we get 2nd conditions. we have

$$\frac{\partial U}{\partial t} = g(r)$$

taking H.T of both sides we have

$$\int_0^\infty \frac{\partial U}{\partial t} r J_0(pr) dr = \int_0^\infty g(r) r J_0(pr) dr$$

$$\boxed{\frac{d\tilde{U}}{dt} = \tilde{g}(p)} \quad \text{———— (4)}$$

Now the solution of (1) we get

$$(D^2 + p^2 c^2)\tilde{U} = 0$$

$$\tilde{U} = A \cos pct + B \sin cpt \quad \text{———— (5)}$$

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when $t = 0$, we have

$\bar{f}(p) = A$ using (3)

~~then~~

$\frac{d\bar{u}}{dt} = -Acp \sin cpt + cpB \cos cpt$

~~at~~ $t = 0$

$\bar{g}(p) = cpB$

$\Rightarrow$ $B = \frac{\bar{g}(p)}{cp}$

then we have from (5)

$\bar{u} = \bar{f}(p) \cos pct + \frac{\bar{g}(p)}{pc} \sin pct$

taking inversion formula of Hankel transformation we have

$$u(r,t) = \int_0^\infty \bar{f}(p) \cos pct \; p J_0(pr) \, dp + \frac{1}{c}\int_0^\infty \frac{\bar{g}(p)}{~~pc~~} \sin pct \; ~~p~~ J_0(pr) \, dp$$

(Proved)

2) Find the potential $V(r,z)$ due to a flat circular disk of unit radius with the centre at the origin and axis along the z axis satisfying the differential equation.

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0,$$

$0 \le r < \infty$

and bdd condition $z \geq 0$

$V = V_0$ when $z = 0$, $0 \le r < 1$

$\frac{\partial V}{\partial z} = 0$ when $z = 0$, $r > 1$

Soln

The given differential equation -

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0$$

taking H.T on both sides of $n=0$,

$$\int_0^\infty \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right) r J_0(pr)\, dr + \int_0^\infty \frac{\partial^2 V}{\partial z^2} r J_0(pr)\, dr$$

$$\Rightarrow -p^2 \tilde{V} = -\frac{\partial^2}{\partial z^2}\int_0^\infty V r J_0(pr)\, dr$$

$$= -\frac{d^2 \tilde{V}}{dz^2}$$

$$\Rightarrow \frac{d^2 \tilde{V}}{dz^2} - p^2 \tilde{V} = 0 \quad \text{--- (1)}$$

$\left[\text{where } \tilde{V} = \int_0^\infty V r J_0(pr)\, dr\right]$

The g.s of (1) given by

$$\tilde{v} = Ae^{pz} + Be^{-pz}$$

now $\tilde{v}$ must vanish when $z \to \infty$ and in this case we must have $A = 0$,

then

$$\tilde{v} = Be^{-pz}$$

applying inversion formula we get —

$$v(r, z) = \int_0^\infty B(p)\, e^{-pz}\, p\, J_0(pr)\, dp$$

[B is a func of p only as it is independent of z]

when $z = 0$, $v = v_0$

$$v_0 = \int_0^\infty B(p)\, p\, J_0(pr)\, dr$$

$0 \leq r \leq 1$

now

$$\frac{\partial v}{\partial z} = \int_0^\infty B(p) \frac{\partial (e^{-pz})}{\partial z}\, p\, J_0(pr)\, dp$$

$$= -\int_0^\infty p^2 B(p)\, e^{-pz}\, J_0(pr)\, dp$$

when $z = 0$, $\frac{\partial v}{\partial z} = 0$

$$\Rightarrow \int_0^\infty p^2 B(p)\, J_0(pr)\, dp = 0$$

$r > 1$

Thus we have

$$\left.\begin{aligned} &\int_0^\infty p\, B(p)\, J_0(pr)\, dp = V_0\,, \quad 0 \le r < 1 \\ &\text{and} \\ &\int_0^\infty p^2 B(p)\, J_0(pr)\, dp = 0 \quad r > 1 \end{aligned}\right\} \text{—(2)}$$

Comparing (2) with the wellknown integrals, we have

$$\left.\begin{aligned} &\int_0^\infty J_0(pr)\, \frac{\sin p}{p}\, dp = \frac{\pi}{2} \quad 0 \le r < 1 \\ &\int_0^\infty J_0(pr)\, \sin p\, dp = 0 \quad r > 1 \end{aligned}\right\} \text{—(3)}$$

where

therefore, $B(p) = \frac{2}{\pi} v_0 \frac{\sin p}{p^2}$

$$\tilde{v} = \frac{2}{\pi} v_0 \frac{\sin p}{p^2} e^{-pz}$$

~~which is required sol~~^{n}.

$$v(r, z) = \int_0^\infty \frac{2}{\pi} v_0 \frac{\sin p}{p^2} e^{-pz}\, p\, J_0(pr)\, dp$$

or $$\frac{d^2\tilde{U}}{dt^2}+c^2p^2\tilde{U}=0 \qquad ...(i)$$

Also when $t=0$.

$$\tilde{U}=\int_0^\infty f(r).r\, J_0(pr)\, dr=\tilde{f}(p) \qquad ...(ii)$$

and $$\int_0^\infty \frac{\partial U}{\partial r}.r\, J_0(pr)\, dr=\int_0^\infty g(r).r\, J_0(pr)\, dr$$

or $$\frac{d\tilde{U}}{dt}=\tilde{g}(p). \qquad ...(iii)$$

Now solution of (i) is given by

$$\tilde{U}=A\cos(pct)+B\sin(pct) \qquad ...(iv)$$

$\therefore$ from (ii) and (iv), $\tilde{f}(p)=A$

and from (iii) and (iv),

$$[-Apc\sin(pct)+Bpc\cos(pct)]_{t=0}=\tilde{g}(p)$$

or $$B=\frac{\tilde{g}[p]}{pc}$$

Hence from (iv), $\tilde{U}=\tilde{f}(p)\cos(pct)+\dfrac{\tilde{g}(p)}{pc}\sin(pct)$.

Applying the inversion formula, we have

$$\tilde{U}(r,t)=\int_0^\infty p\tilde{f}(p)\cos(pct)\, J_0(pr)\, dp$$

$$+\frac{1}{c}\int_0^\infty \tilde{g}(p)\sin(pct)\, J_0(pr)\, dp. \qquad \textbf{Proved.}$$

Ex. 2. *Find the potential $V(r,z)$ of a field due to a flat circular disc of unit radius with the centre at the origin, and axis along the z-axis, satisfying the differential equation*

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0,\ 0\leq r<\infty,\ z\geq 0$$

and the boundary conditions

$$V=V_0 \text{ when } z=0,\ 0\leq r<1$$

and $\dfrac{\partial V}{\partial z}=0$, *when* $z=0,\ r>1$.

Sol. Taking the Hankel transform for $n=0$ of both the sides of the given differential equation, we have

$$\int_0^\infty \left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right) r\, J_0(pr)\, dr=-\int_0^\infty \frac{\partial^2 V}{\partial z^2}.\, J_0(pr)\, dr$$

or $\quad -p^2\widetilde{V} = -\dfrac{d^2\widetilde{V}}{dz^2}$ where $\widetilde{V} = \int_0^\infty V.r\, J_0\,(pr)\, dr$

or $\quad \dfrac{d^2\widetilde{V}}{dz^2} - p^2\,\widetilde{V} = 0$

whose solution is $\widetilde{V}\,(p, z) = A\, e^{pz} + B\, e^{-pz}$.

Now V must vanish as z tends to infinity.

$\therefore \quad \widetilde{V}$ must also vanish as z tends to infinity, $\quad \therefore \quad A = 0$

Hence $\widetilde{V}\,(p, z) = B\, e^{-pz} \qquad B$ is independent of z

i.e. B is a function of p only.

Applying the inversion formula, we have

$$V\,(r, z) = \int_0^\infty B\,(p).e^{-pz}.p\, J_0\,(pr)\, dp \qquad \text{...(i)}$$

$B\,(p)$ is written in place of B to indicate that it depends on p,

Now applying the boundary conditions, we have

when $z = 0, \quad V = \int_0^\infty p\; B\,(p)\; J_0\,(pr)\; dp = V_0,\; 0 \leqslant r < 1$

and $\left(\dfrac{\partial V}{\partial z}\right)_{z=0} = \int_0^\infty -p^2\, B\,(p)\; J_0\,(pr)\; dp = 0,\; r > 1$

$$\left.\begin{aligned} \text{or} \quad & \int_0^\infty p\; B\,(p)\; J_0\,(pr)\; dp = V_0,\; 0 \leqslant r < 1 \\ \text{and} \quad & \int_0^\infty p^2\, B\,(p)\, J_0\,(pr)\, dp = 0,\; r > 1 \end{aligned}\right\} \qquad \text{...(ii)}$$

Comparing (ii) with the well known integrals

$$\left.\begin{aligned} & \int_0^\infty J_0\,(pr).\frac{\sin p}{p}\; dp = \frac{\pi}{2},\; 0 \leqslant r < 1 \\ \text{and} \quad & \int_0^\infty J_0\,(pr).\sin\, p\; dp = 0,\; r > 1 \end{aligned}\right] \qquad \text{...(iii)}$$

we have, $\quad B\;(p) = \dfrac{2}{\pi}\; V_0\; \dfrac{\sin p}{p^2}$.

Hence from (i), $V\,(r, z) = \dfrac{2V_0}{\pi} \displaystyle\int_0^\infty e^{-pz}.\frac{\sin p}{p}.J_0\;(pr)\; dp$.

Ex. 3. *Heat is supplied at a constant rate Q per unit area per unit time over circular area of radius a in the plane $z = 0$ to an infinite solid of conductivity k. Show that the steady temperature at a point distant r from the axis of the circular area and distant z from the plane $z = 0$ is given by*

$$\frac{Qa}{2k}\cdot\int_0^{\infty} e^{-pz}\, J_0(pr)\, J_1(pa).p^{-1}\, dp.$$

Sol. Let $V(r, z)$ be the temperature at the point (r, z) then it is governed by the differential equation

$$\frac{\partial V}{\partial t}=k\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}\right)$$

Since the temperature is steady, $\frac{\partial V}{\partial t}=0$.

Thus the steady temperature at any point (r, z) is given by

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0 \qquad \text{...(i)}$$

with the boundary conditions

$$2\left(-k\frac{\partial V}{\partial z}\right)=Q,\ 0 \leq r < a,\ z=0$$

and $$2.\left(-k\frac{\partial V}{\partial z}\right)=0,\ r > a,\ z=0.$$

There is symmetry about the plane $z=0$, so we consider the temperatures only in the case $z > 0$.

Taking the Hankel transform for $n=0$ of both the sides of equation (i) and the boundary conditions we have

$$\int_0^{\infty}\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right) r\, J_0(pr)\, dr+\int_0^{\infty}\frac{\partial^2 V}{\partial z^2}.r\, J_0(pr)\, dr=0$$

or $$-p^2\widetilde{V}+\frac{d^2\widetilde{V}}{dz^2}=0 \qquad \text{...(ii)}$$

where $$\widetilde{V}=\int_0^{\infty} V.r\, J_0(pr)\, dr$$

and $$\frac{d\widetilde{V}}{dz}=\int_0^{a}\left(-\frac{Q}{2k}\right) r\, J_0(pr)\, dr+\int_a^{\infty} 0.r\, J_0(pr)\, dr$$

when $z=0$

or $$\left(\frac{d\widetilde{V}}{dz}\right)_{z=0}=-\frac{Q}{2k}\int_0^{a}\frac{1}{p}\frac{d}{dr}\{r\, J_1(pr)\}\, dr$$

writing pr for x
and $n=1$, in Rec. formula (vi)
$\frac{1}{p}\frac{d}{dr}\{rJ_1(pr)\}=rJ_0(pr)$
since $J_1(0)=0$

$$\left(\frac{\partial \widetilde{V}}{\partial z}\right)_{z=0}=-\frac{Q}{2k}\cdot\frac{a}{p}\, J_1(ap) \qquad (3)$$